Some Design Approaches of Fractional Order Controllers for Electro-mechanical and Automotive Systems

Guido Maione

Dept. of Electrical and Information Engineering
Polytechnic University of Bari, Italy

Cost Training School
“Fractional order control: theory and applications”
Catania, Italy, September 8, 2017
Outline

1 Introduction
   - Background
   - Motivation

2 Symmetrical Optimum
   - The method by integer-order controllers
   - Extension to fractional-order controllers

3 Realization

4 Loop Shaping
   - The fundamental idea
   - Plant without deadtime
   - Plant with deadtime

5 Loop Shaping & D-decomposition
Background on Fractional Order Controllers

- Non-integer-order controllers: Bode, Tustin, Manabe, Yoshida, Carlson & Halijak
- CRONE controllers (Oustaloup)
- TID controller (Lurie)
- $PI^\lambda D^\mu$ controllers (Podlubny)
- FO-PID, FO-PI, FO-PD controllers
- Fractional order controllers (FOC)
- Fractional order lead/lag networks
- etc. ... ...
Motivation of this lecture

Show application of FOC, namely FO-PI controllers, to:
– mechatronic and automotive systems
– industrial systems
to prove usefulness and benefits of FOC w.r.t. PI in wide-spread and/or important applications

PI/PID are the most applied controllers in these contexts: they are in more than 90% of the industrial control loops\(^1\)

In industry, tuning rules are often the same, widely accepted ones: for example, Ziegler-Nichols, optimum modulus criterion or symmetrical optimum method

Then FO-PI and similar/extended design/tuning rules

---

Methods, controlled systems, fractional orders of controllers

- **SOM**
  \[ G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad 0 < \alpha < 1 \]

- **LS CASE 1**
  \[ G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad 0 < \alpha < 1 \]

- **LS CASE 2**
  \[ G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad 1 < \alpha < 2 \]

- **LS CASE 3**
  \[ G_p(s) = \frac{K_e}{1 + T_e s} \quad 1 < \alpha < 2 \]

- **LS CASE 4**
  \[ G_p(s) = \frac{K_e}{1 + T_e s} e^{-\tau s} \quad 1 < \alpha < 2 \]

- **LS CASE 5**
  \[ G_p(s) = \frac{K_e}{s(1 + T_e s)} e^{-\tau s} \quad 0 < \alpha < 1 \]

- **LS CASE 6**
  \[ G_p(s) = \frac{K_e}{s(1 + T_e s)} e^{-\tau s} \quad 1 < \alpha < 2 \]

The approaches can be extended to other plant models.
Symmetrical Optimum Method:
The method by integer-order controllers
Recall the old Bode’s idea

A reference optimal loop response\(^2\):

- The controller design mainly consists in shaping the asymptotic gain diagram, mainly in choosing the slope of the segment crossing the frequency axis.
- The gain diagram must maintain this slope in a wide frequency interval around the crossover.

The phase margin is constant in the same interval and stability robustness is guaranteed even for high gain variations.

\[
G_{\text{Bode}}(s) = \left( \frac{\omega gc}{s} \right)^\alpha
\]

The standard *Symmetrical Optimum Method* (SOM)

We take inspiration by the SOM, which is common for thermal & electro-mechanical plants:

- Phase margin equal to the specified $PM \approx 37^\circ$ is obtained at $\omega_{max} \equiv \omega_{gc}$, where $\omega_{max}$ is the freq. at which maximum margin can be reached and $\omega_{gc}$ is the gain crossover freq.

- Bode diagram (magnitude & phase) symmetrical w.r.t. $\omega_{gc}$
The standard Symmetrical Optimum Method (SOM)

\[ G_{pl}(s) = k_{pl} \frac{\prod_j (1 + \tau_j s)}{s \prod_i (1 + \tau_i s)} e^{-L s} \approx \frac{K_e}{s(1 + T_e s)} \]

\[ K_e = k_{pl} \quad T_e = \sum_i \tau_i - \sum_j \tau_j + L \]

\[ G_c(s) = K_P \left(1 + \frac{1}{T_I s}\right) \]

\[ K_P = \frac{1}{2 K_e T_e} \]

\[ T_I = 4 T_e \]

\[ G(s) = G_c(s) G_{pl}(s) = \frac{1 + 4 T_e s}{8 T_e^2 s^2 (1 + T_e s)} \]
The standard **Symmetrical Optimum Method (SOM)**
The standard *Symmetrical Optimum Method* (SOM)

\[ w_{b1} = \frac{1}{T_L} = \frac{1}{(4 \ T_e)} \]

\[ w_{b2} = \frac{1}{T_e} \]

\[ w_c = 4.03 \text{ rad/s} = \frac{1}{(2 \ T_e)} \]
The standard **Symmetrical Optimum Method (SOM)**
The standard **Symmetrical Optimum Method (SOM)**

\[ G(s) = G_c(s) \ G_p(s) = \frac{1 + 4 \ T_e \ s}{8 \ T_e^2 \ s^2 \ (1 + T_e \ s)} \]

\[ G_{closed}(s) = \frac{1 + 4 \ T_e \ s}{1 + 4 \ T_e \ s + 8 \ T_e^2 \ s^2 + 8 \ T_e^3 \ s^3} \]

**Step response:**
- rise time (100\%): \( t_r = 3.1 \ T_e \)
- settling time (2\%): \( t_s = 16.5 \ T_e \)
- max overshoot: \( \text{OS}\% = 43.4\% \)

**Smoothing pre-filter:** \( G_F(s) = \frac{1}{1 + \tau s} \quad \tau \in [4 \ T_e, 5 \ T_e] \)

Problem of SOM: high sensitivity to variations of plant gain
Applicability of SOM

Not only for

\[ G_{pl1}(s) = k_{pl} \frac{\prod_j (1 + \tau_j s)}{s \prod_i (1 + \tau_i s)} e^{-Ls} \]

but also for

\[ G_{pl2}(s) = \frac{\bar{k}_{pl}}{(1 + T s)} \frac{\prod_j (1 + \tau_j s)}{\prod_i (1 + \tau_i s)} e^{-Ls} \]

if \( T \gg T_e \) and \( T \gg \tau_i \), by taking:

\[ K_e = k_{pl} = \frac{\bar{k}_{pl}}{T} \]
\[ T_e = \sum_i \tau_i - \sum_j \tau_j + L \]
Symmetrical Optimum

Symmetrical Optimum Method:

Extension to fractional order controllers
Tuning of PI$^\alpha$ controllers

\[ G_c(s) = K_P \left( 1 + \frac{1}{(T_i s)^\alpha} \right) \quad 0 < \alpha < 1 \]

Basic idea: choose gain crossover freq. so that phase margin is maximum (then max phase, i.e. \( \frac{d}{d\omega} \arg(G(j\omega)) = 0 \))

- Analytical solution of a non-linear function minimization problem to obtain the tuning formulae
- Formulae take advantage of \( \alpha \) to offer an excellent trade-off between dynamic performance and stability robustness
Tuning of $\text{PI}^\alpha$ controllers

Solution to the problem:

$C = 1 + \cos(0.5\pi \alpha), \ S = \sin(0.5\pi \alpha)$

$\theta_a = \tan(0.5\pi - 0.5\pi \alpha), \ \theta_b = \tan(PM_\alpha)$

$a = \sqrt{\frac{C (1 + \theta_a \theta_b) - S (\theta_a - \theta_b)}{C (\theta_a - \theta_b) + S (1 + \theta_a \theta_b)}}$

$PM_\alpha = \tan^{-1}\left(\frac{S}{C}\right) - \tan^{-1}\left(a^{-2}\right) + 0.5 \pi (1 - \alpha)$

is the maximum phase margin
Tuning of PI^{\alpha} controllers

\[ a = a(\alpha), \ C = C(\alpha) \]

Formulae take advantage of \( \alpha \) to offer an excellent trade-off between dynamic performance and stability robustness:

\[ T_I = a^2 T_e \]

\[ K_P = \frac{1}{K_e T_I} \sqrt{\frac{1 + a^4}{2 a^4 C}} \]

Note that classical SOM with phase margin specification \( PM \approx 37^\circ \) gives:

\[ a = \frac{1 + \sin(PM)}{\cos(PM)} = 2.0057 \approx 2 \]

\[ T_I = a^2 T_e = 4 T_e \quad K_P = \frac{a}{K_e T_I} = \frac{1}{2 K_e T_e} \]
Realization

Open a parenthesis ...
Realization

$s^\alpha$, with $\alpha \geq 0$, is infinite-dimensional, irrational $\Rightarrow$ realization requires approximation by a rational transfer function of order $N$

There are many different methods (e.g. ORA)

Requirement for effective control:

- Stability: negative real poles
- Minimum-phase: negative real zeros
- Interlacing between zeros and poles
- Low order $N$
Oustaloup Recursive Approximation (ORA)

Operator of non-integer order $\alpha$

$$s^\alpha \approx k \prod_{i=1}^{N} \frac{1 + s/\omega_{z_i}}{1 + s/\omega_{p_i}}$$

$$\omega_{z_1} < \omega_{p_1} < \omega_{z_2} < \omega_{p_2} < \ldots < \omega_{z_N} < \omega_{p_N}$$

Choose the range $[\omega_L, \omega_H]$ in which to achieve the ORA

$$\zeta = \left(\frac{\omega_H}{\omega_L}\right)^{\frac{\alpha}{N}} \quad \eta = \left(\frac{\omega_H}{\omega_L}\right)^{\frac{1-\alpha}{N}}$$

$$\omega_{z_1} = \omega_L \sqrt{\eta}$$

$$\omega_{p_i} = \omega_{z_i} \zeta \quad i = 1, \ldots, N$$

$$\omega_{z_{i+1}} = \omega_{p_i} \eta \quad i = 1, \ldots, N - 1$$

Zeros and poles are negative real and *interlaced*
CFE approximation

CFE is a *continued fraction expansion*. What is it?

Example:

\[ e = \sum_{i=0}^{\infty} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots = 2 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \ldots}}} \]

Truncated CFE:

\[ e \approx 2 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \ldots + \frac{n+1}{n+1}}} \ldots} \]
CFE approximation – The convergent

A CFE of a number or function $f$ is defined by partial numerators $a_j \neq 0$ and partial denominators $b_j$:

$$f = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 + b_2} + \frac{a_3}{b_1 + b_2 + b_3} + \cdots$$

The truncated CFE determines the $n$-th convergent:

$$f \approx f_n = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 + b_2} + \frac{a_3}{b_1 + b_2 + b_3} + \cdots + \frac{a_n}{b_1 + b_2 + \cdots + b_n}$$

$$f = e \approx 2.71828182845905:$$
$$f_0 = 2, f_1 = 3, f_2 = 2.6667, f_3 = 2.7273, f_4 = 2.7170, f_5 = 2.7184, \ldots$$
CFE approximation – General binomial function
(Lorentzen & Waadeland)

\[ f = (1 + x)^\alpha = 1 + \frac{\alpha x}{1 + (1 - \alpha) x} \]
\[ 1 + \frac{(1 + \alpha) x}{2 + (2 - \alpha) x} \]
\[ 3 + \frac{(2 + \alpha) x}{2 + (2 + \alpha) x} \]
\[ 5 + \frac{(3 - \alpha) x}{2 + (3 - \alpha) x} \]
\[ 7 + \ldots \]

\( b_0 = 1 \)
\( a_1 = \alpha x \quad b_1 = 1 \)
\( a_j = (k - \alpha) x \quad b_j = 2 \)
\( a_{j+1} = (k + \alpha) x \quad b_{j+1} = j + 1 \quad j = 2k, \ k \geq 1 \)
CFE approximation – Recurrent computation

\[ f \approx f_n = \frac{P_n(\alpha, x)}{Q_n(\alpha, x)} \]

Determination by recurrence relations:

\[ P_{-1}(\alpha, x) = P_0(\alpha, x) = 1 \]

\[ P_n(\alpha, x) = b_n P_{n-1}(\alpha, x) + a_n P_{n-2}(\alpha, x) \quad n \geq 1 \]

\[ Q_{-1}(\alpha, x) = 0, Q_0(\alpha, x) = 1 \]

\[ Q_n(\alpha, x) = b_n Q_{n-1}(\alpha, x) + a_n Q_{n-2}(\alpha, x) \quad n \geq 1 \]

CFE allows convergence in wider region than PSE
CFE approximation – Examples

\( n = 1 : f = (1 + x)^\alpha \approx f_1 = \frac{P_1(\alpha, x)}{Q_1(\alpha, x)} = 1 + \alpha x \)

\( n = 2 : f = (1 + x)^\alpha \approx f_2 = \frac{P_2(\alpha, x)}{Q_2(\alpha, x)} = \frac{2 + (1 + \alpha) x}{2 + (1 - \alpha) x} \)

\( n = 4 : f = (1 + x)^\alpha \approx f_4 = \frac{P_4(\alpha, x)}{Q_4(\alpha, x)} = \frac{12 + (12 + 6 \alpha) x + (2 + 3 \alpha + \alpha^2) x^2}{12 + (12 - 6 \alpha) x + (2 - 3 \alpha + \alpha^2) x^2} \)

\( n = 6 : f \approx f_6 = \frac{120 + (180 + 60 \alpha) x + (72 + 60 \alpha + 12 \alpha^2) x^2 + (6 + 11 \alpha + 6 \alpha^2 + \alpha^3) x^3}{120 + (180 - 60 \alpha) x + (72 - 60 \alpha + 12 \alpha^2) x^2 + (6 - 11 \alpha + 6 \alpha^2 - \alpha^3) x^3} \)

If \( n \) is odd, the numerator has higher degree than denominator.
Now put $x = (s - 1)$ in $f = (1 + x)^\alpha$ and stop the CFE at the $2N$-th convergent to obtain the $N$-order approximation of $s^\alpha$

$$s^\alpha \approx \frac{\rho_{N,0} s^N + \rho_{N,1} s^{N-1} + \ldots + \rho_{N,N-1} s + \rho_{N,N}}{q_{N,0} s^N + q_{N,1} s^{N-1} + \ldots + q_{N,N-1} s + q_{N,N}}$$

$$\rho_{N,j} = q_{N,N-j} = (-1)^j \binom{N}{j} (\alpha + j + 1)(N-j) (\alpha - N)(j)$$

Pochammer functions:

$$(\alpha + j + 1)(N-j) = (\alpha + j + 1)(\alpha + j + 2) \ldots (\alpha + N)$$

$$(\alpha - N)(j) = (\alpha - N)(\alpha - N + 1) \ldots (\alpha - N + j - 1)$$

$$(\alpha + j + 1)(0) = (\alpha - N)(0) = 1$$
CFE approximation – Examples of Poch. functions

\[(\alpha + j + 1)_{(N-j)} = (\alpha + j + 1)(\alpha + j + 2) \ldots (\alpha + N) \quad (N-j) \text{ factors}\]
\[(\alpha - N)_{(j)} = (\alpha - N)(\alpha - N + 1) \ldots (\alpha - N + j - 1) \quad j \text{ factors}\]

\(N = 1\)

\(j = 0 : (\alpha + j + 1)_{(1-j)} = (\alpha + 1)_{(1)} = (\alpha + 1)\)
\(j = 1 : (\alpha + j + 1)_{(1-j)} = (\alpha + 2)_{(0)} = 1\)
\(j = 0 : (\alpha - 1)_{(0)} = 1\)
\(j = 1 : (\alpha - 1)_{(1)} = (\alpha - 1)\)

\(N = 2\)

\(j = 0 : (\alpha + 1)_{(2)} = (\alpha + 1)(\alpha + 2)\)
\(j = 1 : (\alpha + 2)_{(1)} = (\alpha + 2)\)
\(j = 2 : (\alpha + 3)_{(0)} = 1\)
\(j = 0 : (\alpha - 2)_{(0)} = 1\)
\(j = 1 : (\alpha - 2)_{(1)} = (\alpha - 2)\)
\(j = 2 : (\alpha - 2)_{(2)} = (\alpha - 2)(\alpha - 1)\)
CFE approximation – Examples for varying order $N$

\begin{align*}
N = 1 : \quad s^\alpha & \approx \frac{(1+\alpha)s+(1-\alpha)}{(1-\alpha)s+(1+\alpha)} \\
N = 2 : \quad s^\alpha & \approx \frac{(\alpha^2+3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2-3\alpha+2)}{(\alpha^2-3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2+3\alpha+2)} \\
N = 3 : \quad s^\alpha & \approx \frac{(\alpha^3+6\alpha^2+11\alpha+6)s^3+(3\alpha^3-6\alpha^2-27\alpha+54)s^2+(3\alpha^3-6\alpha^2-27\alpha+54)s+(\alpha^3+6\alpha^2-11\alpha+6)}{(-\alpha^3+6\alpha^2-11\alpha+6)s^3+(3\alpha^3-6\alpha^2-27\alpha+54)s^2+(3\alpha^3-6\alpha^2-27\alpha+54)s+(\alpha^3+6\alpha^2+11\alpha+6)}
\end{align*}

\[G_{\text{CFE}}(\alpha, s) = \tilde{k} \prod_{i=0}^{N} \frac{1 + s/\tilde{\omega}_z_i}{1 + s/\tilde{\omega}_p_i}\]

with interlaced negative real singularities

\[\tilde{\omega}_z_1 < \tilde{\omega}_p_1 < \tilde{\omega}_z_2 < \tilde{\omega}_p_2 < \ldots < \tilde{\omega}_z_N < \tilde{\omega}_p_N\]
Remark 1

Since results are obtained for a fractional order $0 < \alpha < 1$, the approximation of $s^\alpha$ for $1 < \alpha < 2$ is obtained by taking $s^\alpha = s \, s^{\alpha-1}$ and approximating $s^{\alpha-1}$.

Remark 2

If we approximate $f(s) = (1 + s)\alpha$ by a truncated CFE and hence apply the transformation $s = -\frac{2}{z+1}$, we get a stable, minimum-phase Digital Fractional Order Differentiator useful for DSP applications.

Example: for $n = 8 \Rightarrow N = 4$:

$$D_4(z) = \left(\frac{2}{T}\right)^\alpha \frac{1680 \, z^4 - 1680\alpha \, z^3 + (720\alpha^2 - 1440) \, z^2 + (-160\alpha^3 + 880\alpha) \, z + (16\alpha^4 - 160\alpha^2 + 144)}{1680 \, z^4 + 1680\alpha \, z^3 + (720\alpha^2 - 1440) \, z^2 - (-160\alpha^3 + 880\alpha) \, z + (16\alpha^4 - 160\alpha^2 + 144)}$$
As $1/T$ increases, discrete $z$-transfer functions can be very sensitive even to small changes in coefficient values: $z$ & $p$ approach $(1,0)$.

Moreover, since only a finite memory is available, it is impossible to get the values of the coefficients with infinite accuracy.

Hence, as the poles approach to the unity-circle, high-precision representations of the controllers with very long words are necessary to guarantee stability.

But truncation in coefficients is required with finite word length.

Realization based on $\delta$ operator and corresponding complex variable $\gamma$ improves considerably the robustness of the approximation to parameter changes and then to truncation in coefficients.\(^3\)

Benefits of the δ operator

\[ s = \frac{\gamma}{0.5 \gamma T + 1} \]

\[ G_N(s) = \frac{p_{N,0} s^N + p_{N,1} s^{N-1} + \cdots + p_{N,N}}{q_{N,0} s^N + q_{N,1} s^{N-1} + \cdots + q_{N,N}} \implies G_N(\gamma) = \frac{c_{N,0} \gamma^N + c_{N,1} \gamma^{N-1} + \cdots + c_{N,N}}{d_{N,0} \gamma^N + d_{N,1} \gamma^{N-1} + \cdots + d_{N,N}} \]

\[ c_{N,N-j} = \sum_{r=0}^{j} p_{N,N-r} (0.5 T)^{j-r} B(N - r, j - r) \]

\[ d_{N,N-j} = \sum_{r=0}^{j} q_{N,N-r} (0.5 T)^{j-r} B(N - r, j - r) \]

Zeros and poles are all simple, negative real, interlaced, inside stability region for the δ operator.

Implementation with finite word length implies quantization of exact, desired coefficients:

e.g., assuming that a 16-bit processing unit implements the IIR filters by a representation similar to the IEEE 754 32-bit floating point standard format for single precision numbers, then ... ... ...
Benefits of the $\delta$ operator ($\alpha = 0.5, N = 3, T = 0.01$ s)

... see absolute errors between frequency responses obtained by the exact realization and the 16-bit realization

Phase plot of the 16-bit realization of $z$-TF looses flatness at low freq.
Matsuda (Thiele) approximation

The approximating rational TF

\[ G_{\text{Mat}}(s) = \hat{k} \prod_{i=0}^{N} \frac{1 + s/\hat{\omega}_z_i}{1 + s/\hat{\omega}_p_i} \]

with interlaced negative real singularities

\[ \hat{\omega}_z_1 < \hat{\omega}_p_1 < \hat{\omega}_z_2 < \hat{\omega}_p_2 < \ldots < \hat{\omega}_z_N < \hat{\omega}_p_N \]

is obtained by truncating the CFE

\[ f = \alpha_0 + \frac{s - \omega_0}{\alpha_1 + \frac{s - \omega_1}{\alpha_2 + \frac{s - \omega_2}{\alpha_3 + \ldots}}} \]

at the 2N-th convergent
Matsuda approximation

\( \omega_0, \omega_1, \ldots, \omega_{2N} \) are logarithmical spaced frequencies

\( \omega_0, \omega_{2N} \) are chosen so that \( \hat{\omega}_{z_1} \) and \( \hat{\omega}_{p_N} \) are nearly the same as the corresponding \( \tilde{\omega}_{z_1} \) and \( \tilde{\omega}_{p_N} \) of the CFE approximation

\( \alpha_0 = \omega_0^\nu \) and for \( k = 1, \ldots, 2N \):

\[
\alpha_k = \frac{\omega_k - \omega_{k-1}}{m_{k-1}(\omega_k) - m_{k-1}(\omega_{k-1})}
\]

\[
m_{k-1}(\omega) = \frac{\omega - \omega_{k-1}}{m_{k-1}(\omega) - m_{k-1}(\omega_{k-1})}
\]

Remark: the same choice is done for the ORA, i.e. the range \([\omega_L, \omega_H]\) is chosen so as to achieve that \( \omega_{z_1} \) and \( \omega_{p_N} \) are nearly the same as in CFE approximation
Frequency response of $s^{0.5}$ with $N = 4$: Magnitude

- **Oustaloup**: $k = 0.1377$
  - Zeros: -0.0311 -0.2261 -1.6419 -11.9237
  - Poles: -0.0839 -0.6093 -4.4247 -32.1323

- **Maione (CFE)**: $k = 0.1111$
  - Zeros: -0.0311 -0.3333 -1.4203 -7.5486
  - Poles: -0.1325 -0.7041 -3.0000 -32.1634

- **Matsuda**: $k = 0.1109$
  - Zeros: -0.0310 -0.3327 -1.4211 -7.5702
  - Poles: -0.1321 -0.7035 -3.0055 -32.2772

\(\alpha\) is denoted by \(\nu\) in the graphics above.
Frequency response of $s^{0.5}$ with $N = 4$: Phase

Note proximity of Matsuda’s and Maione’s approximation and how they outperform Oustaloup’s.

Note flatness (with $\nu = 0.5$, ideal phase $= \nu \times 90^\circ = 45^\circ$) over a sufficiently wide range.

$\alpha$ is denoted by $\nu$ in the graphics above.
Remark on interlacing

*Interlacing* of **stable** poles & **minimum-phase** zeros is an important property in approximations of fractional operators: most methods achieve interlacing empirically

Instead, the described CFE method formally guarantees interlacing in a systematic way by *closed-form formulas*.

I proved this result formally in the paper:

Interlacing property

- The approximants obtained from 2 different CFEs:
  - a form derived from the Lagrange’s approach, \( f_{2m}^M(s) \)
  - a form obtained by a modified Thiele’s technique, \( f_{2m}^{T2}(s) \)

formally guarantee the interlacing of simple, negative real, zeros and poles, for any \( 0 < \nu < 1 \) and any order \( N \) of the approximation

- The special structure of the 2 truncated CFEs allows to use the Stieltjes’ theorem
The Stieltjes’ result

\[ f^S(s) = 1 + \frac{1}{\alpha_1 s + \frac{1}{\beta_2 + \frac{1}{\alpha_3 s + \frac{1}{\beta_4 + \cdots}}} \]  

Result by Stieltjes (1918) and Gantmacher & Krein (2002)

If the zeros and poles of the rational approximation are all simple, negative real, and interlaced, then \( \alpha_{2k-1} > 0, \beta_{2k} > 0 \).
If \( \alpha_{2k-1} > 0, \beta_{2k} > 0 \), then the interlacing property follows.

Conversion formulas from many CFEs to the Stieltjes’ is unknown and generation of the truncated Stieltjes’ form convergents by repeated division is complex.

Idea is a direct relation between partial num & den of CFEs and \( \alpha_{2k-1} s \) and \( \beta_{2k} \), through variable changes and equivalence transformations.
The 2 results

\[ S^\nu \approx f_{2m}^M(s) = 1 + \frac{\hat{a}_1 (s - 1)}{1 + \frac{\hat{a}_2 (s - 1)}{\ldots + \frac{\hat{a}_{2m} (s - 1)}{1}}} = \frac{\hat{A}_{2m}(\nu, s)}{\hat{B}_{2m}(\nu, s)} \]

\[ \hat{a}_1 = \frac{a_1}{b_1} \quad \text{and} \quad \hat{a}_k = \frac{a_k}{b_{k-1} b_k} \quad \text{for} \quad k \geq 2 \]

\[ S^\nu \approx f_{2m}^{T2}(s) = \hat{c}_0 + \frac{\hat{c}_1 (s - \omega_0)}{1 + \frac{\hat{c}_2 (s - \omega_0)}{\ldots + \frac{\hat{c}_{2m} (s - \omega_0)}{1}}} = \frac{\hat{C}_{2m}(\nu, \hat{x})}{\hat{D}_{2m}(\nu, \hat{x})} \]

Appr. error and its first \(2m - 1\) derivatives are 0 in \(\omega_0\)

\(\hat{c}_k\) computed by closed expressions that use reciprocal differences
Proof is obtained by Stieltjes’ theorem

**Definition**

\[ \hat{A}_{2m}(\nu, s) = K_A \prod_{i=1}^{m} (s + \mu_i^s) \quad \text{and} \quad \hat{B}_{2m}(\nu, s) = K_B \prod_{i=1}^{m} (s + \lambda_i^s) \]

are called a **positive pair** of degree \( m \) iff:

\[ 0 < \mu_1^S < \lambda_1^S < \mu_2^S < \lambda_2^S < \cdots < \mu_m^S < \lambda_m^S \quad \text{and} \quad K_A > 0, K_B > 0 \]

**Theorem**

*For every positive pair of degree \( m \geq 1 \), there exists one, and only one, CFE of the form given by:*

\[
f_{2m}^S(s) = 1 + \frac{1}{\alpha_1 s + \frac{1}{1 + \frac{1}{\alpha_2 s + \frac{1}{\cdots + \frac{1}{\alpha_{2m-1} s + \frac{1}{\beta_{2m}}}}}}}.
\]

*always with \( \alpha_{2k-1} > 0, \beta_{2k} > 0 \), for \( k = 1, 2, \ldots, m \) (Stieltjes, 1918)*
Application 1: Position control of DC motor by SOM

A permanent magnet brushed DC motor is a relatively easy system to model and control but can be used as a case-study and to develop new control strategies.

The motor is modeled as a first-order system \((K_e, T_e)\) with an integrator \(1/s\) and no deadtime \((L = 0)\):

\[
G_{pl}(s) = \frac{K_e}{s(1 + T_e s)}
\]
Simulation & experiments

- The plant is a non-linear 370W dc servomotor
- A power amplifier drives the plant
- A PC equipped with a dSPACE board acquires data, provides the position reference and runs the controller in Matlab/Simulink

\[ G_{pl}(s) = \frac{0.935}{s(1 + 0.124s)} \]
Simulation & experiments: Step response

Good agreement sim-exp

$\alpha = 1$: PI with SOM & smoothing pre-filter
Simulation & experiments: Open-loop freq. response

No more symmetry around crossover but phase nearly flat in a wide range around it.

Very good trade-off between performance & robustness.
### Simulation & experiments: Performance indices

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>OS%</th>
<th>$t_r / T_I$</th>
<th>$t_s / T_I$</th>
<th>$PM_\alpha[^{\circ}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sim</td>
<td>Exp</td>
<td>Sim</td>
<td>Exp</td>
</tr>
<tr>
<td>0.2</td>
<td>2.97</td>
<td>2.32</td>
<td>2.60</td>
<td>2.91</td>
</tr>
<tr>
<td>0.3</td>
<td>6.09</td>
<td>6.27</td>
<td>2.25</td>
<td>2.27</td>
</tr>
<tr>
<td>0.4</td>
<td>10.3</td>
<td>10.7</td>
<td>2.03</td>
<td>2.00</td>
</tr>
<tr>
<td>0.5</td>
<td>14.8</td>
<td>14.6</td>
<td>1.85</td>
<td>1.83</td>
</tr>
<tr>
<td>0.6</td>
<td>21.6</td>
<td>21.3</td>
<td>1.74</td>
<td>1.71</td>
</tr>
<tr>
<td>0.7</td>
<td>28.4</td>
<td>28.7</td>
<td>1.62</td>
<td>1.60</td>
</tr>
<tr>
<td>1 (SOM)</td>
<td>43.4</td>
<td>43.8</td>
<td>0.53</td>
<td>0.48</td>
</tr>
<tr>
<td>1 (SOM &amp; p.f.)</td>
<td>8.10</td>
<td>5.97</td>
<td>1.89</td>
<td>1.88</td>
</tr>
</tbody>
</table>

$t_r(0 – 100\%), t_s(2\%)$

Proper choice of $\alpha$ gives low OS% and excellent $PM$

Drawback of PI$^\alpha$ w.r.t. PI with SOM & pre-filter: settling times
Simulation & experiments: Improving performance

$$\alpha = 0.4, 0.5, 0.6 \quad a_1 = T_I / T_e$$
Main references:

G. Maione, P. Lino  
“New tuning rules for fractional PI$^\alpha$ controllers”  

Maione, G.  
“Continued fractions approximation of the impulse response of fractional order dynamic systems”  
A different approach to approximation: time-domain

Before going to another design method ...
Approximation of the Bode ideal open-loop TF

Obstacles in applying fractional calculus to control problems:

- Study of models in the $t$-domain is difficult
- Simple Laplace inversion formulas from $s$-domain to $t$-domain are lacking
- Complex analytical solutions for impulse/step/ramp responses

$$G(s) = \left( \frac{\omega gc}{s} \right)^{\alpha}$$

Impulse response of non-integer-order integrator:

$$\mathcal{L}^{-1}\{s^{-\alpha}\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

For $\alpha = 1.5$: $g(t) = 2 \sqrt{\frac{t}{\pi}}$
Approximation of the Bode ideal open-loop TF


To approximate the ideal Bode open loop TF:

\[ G_A(s) = \left( \frac{\omega_{gc}}{s + \sigma} \right)^{1.5} \]

\[ \sigma = 0.02 \] to make \( g_A(t) \) square integrable and remove singularity at \( s = 0 \).

Also, \( \sigma \) is chosen s.t. crossover of \( G_A(s) \) is nearly \( \omega_{gc} \) of \( G(s) \) and the same magnitude is obtained in a range around \( \omega_{gc} \).
Impulse response approximation

By a truncated series of Laguerre polynomials:

- Orthogonal functions (available in tables)
- Generalization of exponentials: exp weighted by polynomials
- Very efficient in approximating system dynamics and transient behavior, limited no. of functions, low computational cost

\[ G_B(s) \approx \sum_{i=0}^{N-1} c_i \Lambda_i(k, s) \quad \lambda_i(k, t) = \sum_{j=0}^{i} (-1)^j \binom{i}{i-j} \frac{(k t)^j}{j!} e^{-kt/2} \]

\[ c_i = \int_0^{t_0} e^{-\sigma t} g(t) \lambda_i(k, t) \, dt \quad e^{-\sigma t} g(t) \approx 0 \text{ for } t \geq t_0 \]

\( k \in \mathbb{R} \) is arbitrary chosen to minimize the RMSE in a range around \( \omega_{gc} \)
Plot of impulse responses

A: $\alpha = 1.5$, $\omega_{gc} = 10$ rad/s, $G_A(s) = \left(\frac{10}{s+0.02}\right)^{1.5}$

B: $k = 0.024$, $N = 10$, $G_B(s) \approx \frac{20.316}{s} \frac{s^2+0.274s+0.009}{s^2+0.084s+0.003}$ (stable, min.-phase, low order)
Another t-domain approximation result

G. Maione, “Inverting fractional order transfer functions through Laguerre approximation”, *Systems & Control Letters*, vol. 52, no. 5, 16 Aug. 2004, pp. 387-393, DOI: 0.1016/j.sysconle.2004.02.014

The commensurate FOTF

\[
G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{j=0}^{m} b_j s^j q}{\sum_{i=0}^{n} a_i s^i q}
\]

is expanded in a finite sum of partial fractions of the type \( \frac{K}{s^q + \sigma} \)

Abel integral equation \( \Rightarrow \) Abel FOTF: \( K = 1, \ q = 0.5, \ \sigma = 0.1 \)

\[
\frac{1}{s^{0.5} + 0.1}
\]

We need an approximate inversion
Another approximation result

Approximate inversion by a weighted sum of Laguerre functions of order 1:

\[ g_{NL}(t) = \sum_{i=0}^{N_L} c_i^{(1)} \lambda_i^{(1)}(t) \]

Closed-form expression for computation:

\[ c_i^{(1)} = \frac{\sqrt{2}^\beta}{(i+1)!} \left[ \frac{d^{i+1}}{dz^{i+1}} F(z) \right]_{z=0} \quad \text{with} \quad F(z) = G \left( \frac{\beta^{1+z}}{1-z} \right) \]

\( \beta > 0 \) is a free parameter which is chosen by a systematic method to optimize the approximation.
Another approximation result: $N_L = 4$

$c_0^{(1)} = 0.7714, c_1^{(1)} = -0.1840, c_2^{(1)} = 0.2367, c_3^{(1)} = -0.1025, c_4^{(1)} = 0.1318$

Ideal response $g(t)$: solid line

Laguerre approximation $g_{N_L}(t)$: dashed line
Loop Shaping:
The fundamental idea
Kalman’s optimal feedback system

In a unitary feedback loop, let $G(s)$ be the open-loop TF, then $F(s) = \frac{1}{1+G^{-1}(s)}$ is the closed-loop TF and $1 + G^{-1}(s)$ is the return difference.

According to a seminal paper “a feedback system is optimal if and only if the absolute value of the return difference is at least one at all frequencies”\(^4\):

if this condition occurs, i.e. $|1 + G^{-1}(j\omega)| = 1$, then $y(j\omega) \equiv r(j\omega) \forall \omega$ (perfect I/O tracking), which can’t be for physical systems.

This optimality would imply verifying $|G(j\omega)| \gg 1 \forall \omega$: however high loop gains may lead to instability!

\(^4\)Kalman, R. E., When is a linear control system optimal?” *Trans. ASME, J. Basic Eng.*, Vol. 86, Series D, pp. 84-90 (1964)
Basic idea

**Loop Shaping**

The solution is *shaping* $|G(j\omega)|$ to obtain high loop gains at low frequencies and to make $|G(j\omega)|$ “rolls off” at high frequencies, when stability problems arise.

---


---

A good tracking performance can be ensured within the bandwidth $\omega_B$, that defines $|F(\omega_B)| \approx 1/\sqrt{2}$
Design approach

The controller parameters are selected to obtain:

- desired bandwidth ensuring tracking performance
- guaranteed stable performance despite parameters variations

Procedure:

- first, choose bandwidth and then determine crossover where to guarantee a specified phase margin
- second, take advantage of a fractional integrator so that the plot of \( \text{arg}[G(j\omega)] \) is nearly flat around the crossover (margin does not change within a range around crossover)
Loop Shaping:
Plant without deadtime
CASE 1 – First-order lag plus integrator but no deadtime:

\[ G_p(s) = \frac{K_e}{s(1 + T_e s)} \]

\[ G_c(s) = K_P + \frac{K_I}{s^\nu} = \frac{K_I}{s^\nu} (1 + T_I s^\nu) \quad T_I = \frac{K_P}{K_I} \quad 0 < \nu < 1 \]

Remark on \( 0 < \nu < 1 \). This range allows (see below):

\[ PM = (1 - \nu) \frac{\pi}{2} = 0.5 \left(1 - \nu\right) \pi \]

moreover, the integrator in the plant ensures zero steady-state error to step input \( \Rightarrow \) no integer-order integrator in \( G_c(s) \) unless rejection of plant input disturbance is required (CASE 2)
Design method: mathematical developments

\[ G(j\omega) = G_c(j\omega) G_p(j\omega) = \frac{K_e}{(j\omega)(1 + T_e j\omega)} \frac{K_I}{(j\omega)^\nu} [1 + T_I (j\omega)^\nu] \]

\[ G(j\omega) = \frac{K_e K_I [1 + T_I \omega^\nu (C_1 + jS_1)]}{\omega^{(1+\nu)} (C_2 + jS_2) (1 + j\omega T_e)} \]

\[ C_1 = \cos(0.5 \nu \pi) \quad S_1 = \sin(0.5 \nu \pi) \]

\[ C_2 = \cos(0.5 (1 + \nu) \pi) \quad S_2 = \sin(0.5 (1 + \nu) \pi) \]
Design method: mathematical developments

$$G(j\omega) = \frac{K_e K_I [1 + T_I \omega^\nu (C_1 + jS_1)]}{\omega^{1+\nu} (C_2 + jS_2) (1 + j \omega T_e)}$$

Using $u = \omega T_e$: $G(ju) = \frac{K_e K_I [1 + T_I \left(\frac{u}{T_e}\right)^\nu (C_1 + jS_1)]}{\left(\frac{u}{T_e}\right)^{1+\nu} (C_2 + jS_2) (1 + j u)}$

$$\text{arg}(G(ju)) = \tan^{-1} \left( \frac{T_I \left(\frac{u}{T_e}\right)^\nu S_1}{1 + T_I \left(\frac{u}{T_e}\right)^\nu C_1} \right) - \tan^{-1}(u) - 0.5 (1 + \nu) \pi$$
**Design method: specifications**

1st requirement: bandwidth $u_B$ ensuring a good tracking

Trade-off between fast closed-loop response and need to center $u_C$ in the flat region of the phase diagram

$u_B$ higher than plant bandwidth and s.t. $T_i > 0$ (see below)

By $u_B$ estimation of gain-crossover frequency$^5$:

$$u_C \in \left[ \frac{u_B}{1.7}, \frac{u_B}{1.3} \right] \Rightarrow u_C = \frac{u_B}{1.5}$$

---


Design method: specifications

2nd requirement: phase margin $PM_s$ in a wide range around $u_C$

$$PM_s = \tan^{-1} \left( \frac{T_I \left( \frac{u_C}{T_e} \right)^\nu S_1}{1 + T_I \left( \frac{u_C}{T_e} \right)^\nu C_1} \right) - \tan^{-1}(u_C) + 0.5(1 - \nu)\pi$$

$$= \varphi_1(u_C) - \varphi_2(u_C) + (1 - \nu)\frac{\pi}{2}$$

If possible, select $T_I$ s.t. $\varphi_1(u_C) - \varphi_2(u_C) = 0$. Then:

$$PM_s = (1 - \nu)\frac{\pi}{2} \Rightarrow \nu = 1 - \frac{PM_s}{\pi/2}$$
Design method: relation between $\nu$ and phase margin

$$\nu = 1 - \frac{PM_s}{\pi/2}, \quad PM = (1 - \nu_s) \frac{\pi}{2}$$

A direct relation between the fractional order and the specified phase margin!

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$PM$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>63°</td>
</tr>
<tr>
<td>0.4</td>
<td>54°</td>
</tr>
<tr>
<td>0.5</td>
<td>45°</td>
</tr>
<tr>
<td>0.6</td>
<td>36°</td>
</tr>
</tbody>
</table>
Design method: formula for $T_I$

$$\varphi_1(u_C) - \varphi_2(u_C) = 0$$

$$\Downarrow$$

$$T_I = \frac{(T_e)^\nu}{(u_C)^{\nu-1} (S_1 - u_C C_1)}$$

$$S_1 = \sin(0.5\nu \pi) \quad C_1 = \cos(0.5\nu \pi)$$

Pay care in choosing $u_B$, then $u_C$, and $\nu$ to ensure $T_I > 0$
Design method: formula for $K_I$

Gain crossover: $|G^{-1}(j \, u_C)|^2 = 1$. Then it holds:

$$K_I = \frac{1}{K_e} \left( \frac{u_C}{T_e} \right)^{(1+\nu)} \sqrt{\frac{1 + u_C^2}{1 + 2 \, T_I \left( \frac{u_C}{T_e} \right)^\nu \cdot C_1 + T_I^2 \left( \frac{u_C}{T_e} \right)^{2\nu}}}$$

$$= \frac{1}{K_e} \left( \frac{u_C}{T_e} \right)^{(1+\nu)} \frac{S_1 - u_C \, C_1}{S_1}$$

$$= \frac{1}{K_e} \left( \frac{u_C}{T_e} \right)^{(1+\nu)} \frac{\sin(0.5 \, \nu \, \pi) - u_C \cos(0.5 \, \nu \, \pi)}{\sin(0.5 \, \nu \, \pi)}$$
Introduction Symmetrical Optimum Realization App of SOM Laguerre Loop Shaping LS & D-decomposition Concl

Summary of LS design procedure without deadtime – CASE 1: 1st order lag plus int. and $0 < \nu < 1$

$$G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad G_c(s) = K_P + \frac{K_I}{s^\nu} \quad 0 < \nu < 1$$

1. Set $\nu$. Specifications: $u_B, PM_s$. Then: $u_C$ and

$$\nu = 1 - \frac{PM_s}{\pi/2}$$

2. Set $T_I$:

$$T_I = \frac{(T_e)^{\nu}(u_C)^{(1-\nu)}}{\sin(0.5\nu \pi) - u_C \cos(0.5\nu \pi)}$$

3. Set $K_I$:

$$K_I = \frac{1}{K_e} \left(\frac{u_C}{T_e}\right)^{(1+\nu)} \frac{\sin(0.5\nu \pi) - u_C \cos(0.5\nu \pi)}{\sin(0.5\nu \pi)}$$

Set $K_P = T_I K_I$

4. Approximate $s^\nu$ then $G_c(s) = K_P + \frac{K_I}{s^\nu}$
Application 2: Position control of DC servomotor by LS

The motor is modeled as a first-order system with an integrator and no deadtime

\[ G_p(s) = \frac{0.9779}{s(1+0.0798s)} \]
Application 2: Controller design

\[ u_B = 0.3 \Rightarrow \omega_B = 3.7594 \text{ rad/s}, \quad u_C = u_B/1.7 = 0.1765 \]

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_I )</td>
<td>0.1742</td>
<td>0.1203</td>
<td>0.0968</td>
<td>0.0845</td>
<td>0.0780</td>
</tr>
<tr>
<td>( K_I )</td>
<td>7.9402</td>
<td>8.4690</td>
<td>8.8096</td>
<td>9.0791</td>
<td>9.3173</td>
</tr>
</tbody>
</table>
Application 2: Open-loop frequency response

Look phase margin nearly unchanged despite gain variations
Application 2: Phase margin change

In fact:

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.1u_C$</td>
<td>1.2798</td>
<td>1.1169</td>
<td>0.9553</td>
<td>0.7947</td>
<td>0.6348</td>
</tr>
<tr>
<td>$0.5u_C$</td>
<td>1.2725</td>
<td>1.1129</td>
<td>0.9535</td>
<td>0.7943</td>
<td>0.6352</td>
</tr>
<tr>
<td>$u_C$</td>
<td>1.2567</td>
<td>1.0996</td>
<td>0.9425</td>
<td>0.7854</td>
<td>0.6283</td>
</tr>
<tr>
<td>$2u_C$</td>
<td>1.2209</td>
<td>1.0672</td>
<td>0.9137</td>
<td>0.7605</td>
<td>0.6076</td>
</tr>
<tr>
<td>$5u_C$</td>
<td>1.1078</td>
<td>0.9601</td>
<td>0.8140</td>
<td>0.6700</td>
<td>0.5284</td>
</tr>
</tbody>
</table>
Simulated & experimental step responses

Differences Exp-Sim due to motor static nonlinearities

Increase $\nu$: higher OS%, lower rise time and longer settling time
Comparison: SOM and Loop Shaping

→ classical SOM gives $PM \approx 37^\circ$ at crossover only
→ LS guarantees higher margins in wider ranges

LS improves $t_r$ w.r.t. SOM
SOM with $\alpha = 1$ and pre-filter further improves $t_r$
Application 2: Overshoot

See how overshoot is affected by \( \nu \) and (not!) by \( u_B \)
CASE 2: A different choice with $1 < \nu < 2$

\[
G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad G_c(s) = \frac{K_l}{s^\nu} (1 + T_l s^\nu) \quad 1 < \nu < 2
\]

The controller includes $1/s$ to reject disturbances on the plant input (a torque for a PMSM motor), and the residual non-integer-order integrator is $1/s^{\nu-1} = 1/s^\mu$, $0 < \mu < 1$.

\[
PM_s = \tan^{-1} \left( \frac{T_l \left( \frac{u_C}{T_e} \right)^\nu S_1}{1 + T_l \left( \frac{u_C}{T_e} \right)^\nu C_1} \right) - \tan^{-1}(u_C) + 0.5(1 - \nu)\pi
\]

Select $T_l$ s.t. $\varphi_1(u_C) - \varphi_2(u_C) = \pi/2 \Rightarrow PM_s = (2 - \nu)\pi/2$. 
Summary of LS design procedure without deadtime – CASE 2: 1st order lag plus int. and $1 < \nu < 2$

$$G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad G_c(s) = K_P + \frac{K_I}{s^\nu} \quad 1 < \nu < 2$$

1. Set $\nu$. Specifications: $u_B$, $PM_s$. Then: $u_C$ and $\nu = 2 - \frac{PM_s}{\pi/2}$

2. Set $T_I$:
   $$T_I = \frac{-(T_e)^\nu}{(u_C)^\nu \left[u_C \sin(0.5 \nu \pi) + \cos(0.5 \nu \pi)\right]}$$

3. Set $K_I$:
   $$K_I = \frac{1}{K_e} \left(\frac{u_C}{T_e}\right)^{(1+\nu)} \frac{u_C \sin(0.5 \nu \pi) + \cos(0.5 \nu \pi)}{\sin(0.5 \nu \pi)}$$

Set $K_P = T_I K_I$

4. Approximate $s^\nu$ then $G_c(s) = K_P + \frac{K_I}{s^\nu}$
CASE 3: Slight modification for first-order lag

First-order lag without integrator and deadtime:

\[ G_p(s) = \frac{K_e}{1 + T_e s} \]
\[ G_c(s) = K_P + \frac{K_I}{s^\nu} = \frac{K_I}{s^\nu} (1 + T_I s^\nu) \quad T_I = K_P / K_I \quad 1 < \nu < 2 \]

Remark on \( 1 < \nu < 2 \): controller must add an integer-order integrator, which is not in the plant

\[ G(j\omega) = G_c(j\omega) G_p(j\omega) = \frac{K_e K_I [1 + T_I \omega^\nu (C_1 + jS_1)]}{\omega^\nu (C_1 + jS_1) (1 + j\omega T_e)} \]

\[ PM_s = \tan^{-1} \left( \frac{T_I \left( \frac{u_C}{T_e} \right)^\nu S_1}{1 + T_I \left( \frac{u_C}{T_e} \right)^\nu C_1} \right) - \tan^{-1} (u_C) + (2 - \nu) \pi / 2 \]
Summary of LS design without deadtime – CASE 3: 1st-order lag and $1 < \nu < 2$

$$G_p(s) = \frac{K_e}{1 + T_e s} \quad G_c(s) = K_P + \frac{K_I}{s^\nu} \quad 1 < \nu < 2$$

1. Set $\nu$. Specifications: $u_B$, $PM_s$. Then: $u_C$ and $\nu = 2 - \frac{PM_s}{\pi/2}$

2. Set $T_I$:  
   $$T_I = \frac{(T_e)^\nu (u_C)^{(1-\nu)}}{\sin(0.5\nu \pi) - u_C \cos(0.5\nu \pi)}$$

3. Set $K_I$:  
   $$K_I = \frac{1}{K_e} \left(\frac{u_C}{T_e}\right)^\nu \frac{\sin(0.5\nu \pi) - u_C \cos(0.5\nu \pi)}{\sin(0.5\nu \pi)}$$

   Set $K_P = T_I K_I$

4. Approximate $s^\nu$ then $G_c(s) = K_P + \frac{K_I}{s^\nu}$
References:

P. Lino, G. Maione
“Tuning PI\(\nu\) fractional order controllers for position control of DC-servomotors”

P. Lino, G. Maione
“Loop-shaping and easy tuning of fractional-order proportional integral controllers for position servo systems”

Loop Shaping:

Plant with deadtime
Design method: starting formulas

Let us introduce deadtime or delay in the controlled plant. It is another, very common model in the considered applications

\[ G_p(s) = \frac{K_e}{1 + T_e s} e^{-\tau s} \]

\[ G_c(s) = K_P + \frac{K_I}{s^\nu} = \frac{K_I}{s^\nu} (1 + T_I s^\nu) \quad T_I = \frac{K_P}{K_I} \quad 1 < \nu < 2 \]

Again \( 1 < \nu < 2 \): controller must add an integer-order integrator, which is not in the plant

We will do such that:

\[ PM = (2 - \nu) \frac{\pi}{2} = 0.5 (2 - \nu) \pi \]
Design method: same idea

Approximation of perfect I-O tracking in a desired bandwidth \( u_B \) and ensuring robust stability by appropriately *shaping the open-loop frequency response around the gain crossover frequency*

*Fractional integrator* allows a flat phase diagram in a wide frequency interval

Selection of \( T_i \) and \( \nu \) to get desired phase margin \( PM_s \), held nearly constant in a wide interval around the 0-dB frequency \( u_C \)

A slight modification of the *loop-shaping* procedure to compensate the deadtime
Design method: mathematical developments

\[ G(j\omega) = G_c(j\omega)G_p(j\omega) = \frac{K K_l \left[ 1 + T_l \omega^\nu (C + jS) \right]}{\omega^\nu (C + jS) \left( 1 + j \omega \frac{T_e}{T_e} \right)} e^{-j \omega \tau} \]

\[ C = \cos(0.5 \nu \pi) \quad S = \sin(0.5 \nu \pi) \]

Using \( u = \omega T_e \): \[ G(ju) = \frac{K_e K_l \left[ 1 + T_l \left( \frac{u}{T_e} \right)^\nu (C + jS) \right] e^{-j \frac{u \tau}{T_e}}}{\left( \frac{u}{T_e} \right)^\nu (C + jS) \left( 1 + j u \right)} \]

\[ \arg(G(ju)) = \tan^{-1} \left( \frac{T_l \left( \frac{u}{T_e} \right)^\nu S}{1 + T_l \left( \frac{u}{T_e} \right)^\nu C} \right) - \tan^{-1}(u) - \frac{u \tau}{T_e} - 0.5 \nu \pi \]
Design method: enforcing the specifications

1st specification: bandwidth $u_B$ for a good tracking response

By $u_B$ estimation of gain-crossover frequency: $u_C \in \left[ \frac{u_B}{1.7}, \frac{u_B}{1.3} \right] \Rightarrow u_C = \frac{u_B}{1.5}$

2nd specification: phase margin $PM_s$ in a wide range around $u_C$

\[
PM_s = \tan^{-1} \left( \frac{T_I \left( \frac{u_C}{T_e} \right)^\nu \sin(0.5 \nu \pi)}{1 + T_I \left( \frac{u_C}{T_e} \right)^\nu \cos(0.5 \nu \pi)} \right) - \tan^{-1}(u_C) - \frac{u_C \tau}{T_e} + \pi - \frac{\pi}{2} \nu
\]

\[
= \varphi_1(u_C) - \varphi_2(u_C) - \frac{u_C \tau}{T_e} + \pi - \frac{\pi}{2} \nu
\]

Select $T_I$ s.t. $\varphi_1(u_C) - \varphi_2(u_C) - \frac{u_C \tau}{T_e} = 0$. Then:

\[
PM_s = (2 - \nu) \frac{\pi}{2} \Rightarrow \nu = 2 - \frac{PM_s}{\pi/2}
\]
Design method: formula for $T_I$

\[ \varphi_1(u_C) - \varphi_2(u_C) - \frac{u_C \tau}{T_e} = 0 \]

\[ \Downarrow \]

\[ T_I = \frac{(T_e)^\nu \left[ u_C + \tan \left( \frac{u_C \tau}{T_e} \right) \right]}{(u_C)^\nu \left[ (S - u_C C) - (C + u_C S) \tan \left( \frac{u_C \tau}{T_e} \right) \right]} \]

\[ S = \sin(0.5 \nu \pi) \quad C = \cos(0.5 \nu \pi) \]
Gain crossover: \(|G^{-1}(j\, u_C)|^2 = 1\). Then it holds:

\[
K_I = \frac{1}{K_e} \left( \frac{u_C}{T_e} \right)^\nu \sqrt{\frac{1 + u_C^2}{1 + 2\, T_I \left( \frac{u_C}{T_e} \right)^\nu \, C + T_I^2 \left( \frac{u_C}{T_e} \right)^{2\nu}}}
\]
LS design – CASE 4: 1st-order lag with deadtime

\[ G_p(s) = \frac{K_e}{1 + T_e s} e^{-\tau s} \quad G_c(s) = K_P + \frac{K_I}{s^\nu} \quad 1 < \nu < 2 \]

1. Set \( \nu \). Specifications: \( u_B, PM_s \). Then: \( u_C \) and

\[
\nu = 2 - \frac{PM_s}{\pi/2}
\]

2. Set \( T_I \):

\[
T_I = \frac{(T_e)^\nu \left[ u_C + \tan \left( \frac{u_C \tau}{T_e} \right) \right]}{(u_C)^\nu \left[ S - u_C C - (C + u_C S) \tan \left( \frac{u_C \tau}{T_e} \right) \right]}
\]

3. Set \( K_I \):

\[
K_I = \frac{1}{K_e} \left( \frac{u_C}{T_e} \right)^\nu \sqrt{\frac{1 + u_C^2}{1 + 2 T_I \left( \frac{u_C}{T_e} \right)^\nu C + T_I^2 \left( \frac{u_C}{T_e} \right)^{2\nu}}}
\]

Set \( K_P = T_I K_I \)

4. Approximate \( s^\nu \) then \( G_c(s) = K_P + \frac{K_I}{s^\nu} \)
LS design – CASE 5: 1st-order lag plus int. & deadtime, $0 < \nu < 1$

\[ G_p(s) = \frac{K_e}{s(1 + T_e s)} e^{-\tau s} \quad G_c(s) = K_P + \frac{K_I}{s^\nu} \quad 0 < \nu < 1 \]

1. Set \( \nu \). Specifications: \( u_B, PM_s \). Then: \( u_C \) and \( \nu = 1 - \frac{PM_s}{\pi/2} \)

2. Set \( T_I \):
   \[ T_I = \frac{(T_e)\nu \left[ u_C + \tan\left(\frac{u_C \tau}{T_e}\right)\right]}{(u_C)^\nu \left[ S - u_C C - (C + u_C S) \tan\left(\frac{u_C \tau}{T_e}\right)\right]} \]

3. Set \( K_I \):
   \[ K_I = \frac{1}{K_e} \left(\frac{u_C}{T_e}\right)^{(1+\nu)} \sqrt{\frac{1 + u_C^2}{1 + 2 T_I \left(\frac{u_C}{T_e}\right)^\nu C + T_I^2 \left(\frac{u_C}{T_e}\right)^{2\nu}}} \]
   Set \( K_P = T_I K_I \)

4. Approximate \( s^\nu \) then \( G_c(s) = K_P + \frac{K_I}{s^\nu} \)
LS design – CASE 6: 1st-order lag plus int. & deadtime, $1 < \nu < 2$

$$G_p(s) = \frac{K_e}{s(1 + T_e s)} e^{-\tau s} \quad G_c(s) = K_P + \frac{K_I}{S^\nu} \quad 1 < \nu < 2$$

1. Set $\nu$. Specifications: $u_B$, $PM_s$. Then: $u_C$ and $\nu = 2 - \frac{PM_s}{\pi/2}$

2. Set $T_I$:

$$T_I = \frac{(T_e)^\nu \left[u_C \tan \left(\frac{u_C \tau}{T_e}\right) - 1\right]}{(u_C)^\nu \left[(C + u_C S) + (S - u_C C) \tan \left(\frac{u_C \tau}{T_e}\right)\right]}$$

3. Set $K_I$:

$$K_I = \frac{1}{K_e} \left(\frac{u_C}{T_e}\right)^{(1+\nu)} \sqrt{\frac{1 + u_C^2}{1 + 2 T_I \left(\frac{u_C}{T_e}\right)^\nu C + T_I^2 \left(\frac{u_C}{T_e}\right)^{2\nu}}}$$

Set $K_P = T_I K_I$

4. Approximate $s^\nu$ then $G_c(s) = K_P + \frac{K_I}{S^\nu}$
Application 3: Control of DC motor with deadtime

Time-varying and nonlinear: static friction, dynamic resistance increasing when motion is reverted, nonlinear distortion by brushes ⇒ high starting torque & asymmetrical I/O characteristic

Traditional mathematical models ⇒ FOPTD plant model:

\[ G(s) = \frac{K_e}{s(1 + T_e s)} e^{-\tau s} \quad \text{or} \quad G(s) = \frac{K_e}{1 + T_e s} e^{-\tau s} \]
Application 3: Experimental set-up for DC-motor with deadtime

DC-motor is actuated through a **power amplifier** by means of PC-commands processed and sent by an interface board.

**Encoder** provides the measured position or speed to the board processor, that computes the control action and sends it to the power unit. The board uses 16 bit A/D-D/A converters, provides the position and speed references and runs the controllers in discrete time with the Euler discretization rule.

The controllers are part of a **Simulink block diagram** the board uses to easily and directly control the real plant or its I/O simulation model.

Identified plant parameters: $K_e = 0.9843$, $T_e = 0.0651$ s, $\tau = 0.02$ s

In experiments, the controller is directly applied to the motor by the dSPACE board.
Application 3: Speed control of DC motor with deadtime

\[ K_e = 0.9843, \ T_e = 0.0651 \text{ s, } \tau = 0.02 \text{ s} \]

\[ u_C = 1.8 \]

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( PM_s )</th>
<th>( K_P )</th>
<th>( K_I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>54°</td>
<td>2.5831</td>
<td>148.3770</td>
</tr>
<tr>
<td>1.5</td>
<td>45°</td>
<td>2.9554</td>
<td>289.8783</td>
</tr>
<tr>
<td>1.6</td>
<td>36°</td>
<td>3.5553</td>
<td>563.3830</td>
</tr>
</tbody>
</table>

As basis for comparison, use a PI controller that is tuned by the optimum modulus criterion with a smoothing pre-filter.
Application 3: Speed control of DC motor with $\tau$

Experimental responses to speed step command, speed reversal and to load application
Oscillations increase with $\nu$ (phase margin decreases)

Fractional pre-filters almost cancel the oscillations

Disturbance is better rejected by the FOPI with fractional filters

Performance by PI or FOPI when speed is reverted is comparable because of the opposite action of the brushes when the sense of rotation is not the preferred one.
Application 3: Position control of DC motor with $\tau$
Application 3: Position control of DC motor with $\tau$
Application 4: Speed control of a PMSM motor

A permanent magnet synchronous motor: fast response, high power/weight, reduced size, high efficiency

[Coupling between two PMSMs: one of them is used as a load]
Application 4: Block scheme with PMSM

→ 2 inner control loops for d- and q-axis currents $i_{sd}, i_{sq}$
→ 1 outer control loop for rotor speed $\omega_r$
Application 4: Experimental set-up

\[ T_c = 0.1 \text{ ms: sampling period} \]
Application 4: Inner current loop

Time delays \(\approx\) 1st-order systems with small time constants:

- current sensor \(\tau_L = 0.45\) ms
- signal sampling \(T_c/2\)
- control algorithm \(T_c\)
- holding \(T_c/2\)
- inverter operation \(T_c/2\)
Application 4: Inner current loop

Simplification: unitary feedback loop and $\tau_{\Sigma i} = 5 \frac{T_c}{2} + \tau_L = 0.7 \text{ ms}$

$$G_p(s) = \frac{k_{inv}/R_s}{(1 + \tau_{\Sigma i} s)(1 + T_q s)}$$

$$G_c(s) = \frac{K_{P,isq} (1 + T_{I,isq})}{T_{I,isq} s}$$
Application 4: PI controller for the current

Designed by z-p cancelation and **Optimum Modulus Criterion**:

$$T_{I,isq} = T_q \quad K_{P,isq} = \frac{R_s \ T_{I,isq}}{k_{inv} \ 2 \ \tau_{\Sigma_i}}$$

$$G_c(s) \ G_p(s) = \frac{1}{2 \ \tau_{\Sigma_i} \ s \ (1 + \tau_{\Sigma_i} \ s)} \ \Rightarrow \ G_{cl}(s) = \frac{1}{1 + 2 \ \tau_{\Sigma_i} \ s + 2 \ \tau_{\Sigma_i}^2 \ s^2}$$

<table>
<thead>
<tr>
<th>Characteristics of step resp of $G_{cl}(s)$</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage maximum overshoot</td>
<td>4.3%</td>
</tr>
<tr>
<td>Settling time (2%)</td>
<td>$8.4 \ \tau_{\Sigma_i}$</td>
</tr>
<tr>
<td>Rise time (100%)</td>
<td>$4.7 \ \tau_{\Sigma_i}$</td>
</tr>
</tbody>
</table>

Closed-loop TF of the inner current loop:

$$G_{0,isq}(s) \approx \frac{1}{1 - \tau_L \ s} \ \frac{1}{1 - (T_c/2) s} \ \frac{1}{1 + 2 \ \tau_{\Sigma_i} \ s + 2 \ \tau_{\Sigma_i}^2 \ s^2}$$

where we have approximated $(1 + \tau_L s)$ and $(1 + (T_c/2)s)$ in forward path, in series with an equivalent unitary feedback system.
Application 4: Outer speed loop

Open-loop TF simplified by neglecting $2 \tau^2_{\Sigma i} s^2$ and by summing the time constants ($\tau_{sp}$ for speed sensor):

$$\tau_{\Sigma \omega} = T_c - \tau_L - T_c/2 + 2\tau_{\Sigma i} + \tau_{sp} + T_c/2$$

Then the plant TF in the speed loop is $G_{p,\omega r}(s) = \frac{K_c n_p}{J s (1+\tau_{\Sigma \omega} s)}$
Application 4: PMSM speed control

Combination of FOPI controller and integer/fractional pre-filter is compared with combination of PI controller tuned by the SOM and smoothing pre-filter.

In both cases, the inner current loop includes the PI controller tuned by the Optimum Modulus Criterion.
Application 4: Speed control by PI or FOPI

PI tuned by the Symmetrical Optimum Method:

\[ T_{I,\omega} = 4 \tau \Sigma \omega \quad \quad \quad K_{P,\omega} = \frac{J}{2 \tau \Sigma \omega K_c n_p} \]

\[ G_c(s) G_p(s) = \frac{1+4 \tau \Sigma \omega s}{8 \tau^2 \Sigma \omega s^2 (1+\tau \Sigma \omega s)} \Rightarrow G_{cl}(s) = \frac{1+4 \tau \Sigma \omega s}{1+4 \tau \Sigma \omega s+8 \tau^2 \Sigma \omega s^2+8 \tau^3 \Sigma \omega s^3} \]

FOPI designed by Loop Shaping:

Plant is \( G_{p,\omega r}(s) = \frac{K_c n_p}{J s (1+\tau \Sigma \omega s)} \) : \( K_e = \frac{K_c n_p}{J}, \quad T_e = \tau \Sigma \omega, \quad \tau = 0 \)

\[ u_C = u_B/1.7 \]

<table>
<thead>
<tr>
<th>( u_B )</th>
<th>( u_C )</th>
<th>( \nu )</th>
<th>( PM_s )</th>
<th>( K_P )</th>
<th>( K_I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.02</td>
<td>0.6</td>
<td>1.4</td>
<td>54°</td>
<td>0.1314</td>
<td>5.9296</td>
</tr>
<tr>
<td>1.36</td>
<td>0.8</td>
<td>1.5</td>
<td>45°</td>
<td>0.2004</td>
<td>29.7201</td>
</tr>
<tr>
<td>2.04</td>
<td>1.2</td>
<td>1.6</td>
<td>36°</td>
<td>0.3616</td>
<td>119.5887</td>
</tr>
</tbody>
</table>

\( N = 5 \)
Application 4: Test conditions for PMSM speed control

Step input of $-150 \text{ rad/s}$ is applied at $t = 0.225 \text{ s}$. A load disturbance of 2.2 Nm is superposed at $t = 1.253 \text{ s}$. The motion is reverted at $t = 2.268 \text{ s}$. The load is removed at $t = 3.361 \text{ s}$. 
Application 4: Test conditions for PMSM speed control

FOPI provide a reasonable fast response and small OS% w.r.t. PI, especially with $\nu = 1.6$ and with fractional filters.

PI with a smoothing filter show lower rise times but much more oscillations and settle after slower transients.
Reference:

Loop Shaping & $D$-decomposition

Loop Shaping & $D$-decomposition
Idea of this refined methodology

Combination of:

- **Loop shaping** to design FOPI controllers for reference working points:
  to obtain freq. domain performance specifications for an optimal feedback system and a nearly constant phase margin in a sufficiently wide freq. range

- **D-decomposition** approach to guarantee an enhanced closed-loop robust stability

A *gain scheduling* is used to allow *switching* from one controller to another in order to adapt to various working points
Methodology of $D$-decomposition (1)

$D$-decomposition is a classical approach for robust stability analysis:

*to determine the entire set $\mathcal{D}$ of controller gains leading to a stable closed-loop system:*

- to avoid time-consuming stability checks of new controller settings required for new working conditions
- if the point associated to designed controller gains is far from set boundary, then the stability is still ensured for bounded variations
Methodology of $D$-decomposition (2)

\[ G_p(s) = \frac{K_e e^{-\tau s}}{1 + T_e s} \quad G_c(s) = \frac{K_I (1 + T_I s^\nu)}{s^\nu} \quad G(s) = \frac{K_e K_I (1 + T_I s^\nu) e^{-\tau s}}{(1 + T_e s) s^\nu} \]

\[ F(s) = \frac{K_e K_I (1 + T_I s^\nu) e^{-\tau s}}{(1 + T_e s) s^\nu + K_e K_I (1 + T_I s^\nu) e^{-\tau s}} \]

Roots of the fractional order characteristic pseudo-polynomial equation:

\[ E(s) = (1 + T_e s) s^\nu + K_e K_I (1 + T_I s^\nu) e^{-\tau s} = 0 \]

If all roots lie in the open left-half of the $s$-plane (LHP), then the closed-loop system is BIBO stable.

Set $\mathcal{D}$ of stabilizing controllers: if $(K_P, K_I, \nu) \in \mathcal{D}$ then all roots lie in the LHP.
Methodology of $D$-decomposition (3)

$D$ is defined by the real root boundary (RRB), the infinite root boundary (IRB), and the complex root boundary (CRB)

- RRB comes from solutions of $E(s = 0) = 0$: $K_I = 0$
- IRB for $s \to \infty$ does not exist
- CRB comes from solutions of $E(s = j\omega) = 0$:
  \[
  K_I(\omega) = \frac{\omega^{\nu}(\sin(x)+\omega T_e \cos(x))}{K_e S} \quad K_P(\omega) = \frac{(\omega T_e S - C) \sin(x)-(S+\omega T_e C) \cos(x)}{K_e S}
  \]
  with $C = \cos(0.5 \pi \nu)$, $S = \sin(0.5 \pi \nu)$, $x = \omega \tau$

$\nu$ is fixed and $\omega \in (0, \infty)$: curve in 2D-space ($K_P, K_I$)

$\nu$ is varied: the complete 3D stability domain is obtained
$D$-dec analysis for a representative working point

\[ \times: \text{LS design points for nominal parameters in a working condition} \]

\[ \forall \nu \text{ the system is stable} \]

Changing $u_B$ while keeping $PM_s$ moves $\times$ along the Rel. Stab. Line
$D$-dec analysis for a representative working point

Stability regions for different values of $\nu$

$\forall \nu$ the system is stable

Given $\nu$, $\times$ denotes values of $K_P$ and $K_I$ and corresponds to specified $\omega_c$ and $PM_s$

$\times$ on Rel. Stab. Line given by $PM_s$

Distance between CRB and $\times$-RSL: robustness

Rel. Stab. Line associated to $GM_s$ can be defined: $\times$ corresponds to phase crossover frequency
Application 5: Injection control of CNG engines

**Compressed Natural Gas** automotive engines: low cost, gas distribution, reduction of pollution & consumptions

To keep/increase combustion efficiency, accurate metering of the air-fuel mixture is required

- **control injection timing**: precise adjustment is possible by electro-injectors
- **control injection (rail) pressure** at required levels: challenge

Difficulty is due to gas compressibility:

- working points vary a lot (power and speed requirements)
- complex phenomena & disturbances
CNG injection system

Fuel tank of high pressure gas

Pressure Reducer: a Main Chamber & a Control Chamber

Solenoid Valve regulates flow into CC

Inflow section varied by displacing a Shutter coupled with a Piston

Fuel metering: common rail and 4 electro-injectors

Electronic Control Unit

Injection flow depends on:
- the $p_{rail} \approx p_{MC}$
- the injectors opening time intervals $t_j$ driven by ECU

Tank: 40-200 bar ⇒ Common rail: 5-20 bar
CNG injection system: operation

ECU determines $p_{\text{rail}}$ and controls flow:

1. sets *injection timings* depending on engine speed and load
2. PWM of the duty cycle of the driving current to open/close the SV → CC pressure

Inflow section of the PR is varied by displacing a shutter (S) coupled with a piston (P)

Gas in the MC pushes P up & gas in the CC pushes P down and opens the S

If SV is energized, the fuel enters CC and pushes P down: more fuel enters into the MC, where the pressure increases

If SV is not energized, pressure in CC decreases, P raises and S closes under the action of a preloaded spring
CNG injection system: State-space nonlinear model

- $x_1 = p_{cc}, \ x_2 = p_{rail}$
- $u_1$ input to SV and $u_2$ command to injectors (disturbance)
- $y = x_2$

\[
\begin{align*}
\dot{x}_1(t) &= c_{11} \ p_{tk}(t) \ u_1(t) - c_{12} \ \sqrt{x_2(t)} \ [x_1(t) - x_2(t)] \\
\dot{x}_2(t) &= c_{21} \ p_{tk}(t) \ [c_{24} \ x_1(t) - c_{25} \ x_2(t) - c_{26} \ p_{tk}(t) - c_{27}] - c_{22} \ x_2(t) \ u_2(t) \\
&\quad + c_{23} \ \sqrt{x_2(t)} \ [x_1(t) - x_2(t)]
\end{align*}
\]

Equilibrium: working conditions necessary to inject the proper fuel amount:

- conditions derived from driver power request, engine speed and load
- tank pressure $p_{tk} \approx const$ in a large time interval
- ECU sets $\bar{x}_2$ and $\bar{u}_2$ by means of look-up tables $\Rightarrow \bar{x}_1, \bar{u}_1$ obtained by the model
CNG injection system: linearized model

Input: variation w.r.t. equilibrium of the input to SV
Output: variation w.r.t. equilibrium of rail pressure

Linearization around working points gives:

\[ \frac{K_e}{1 + T_e s} e^{-\tau s} \]

Each \((K_e, T_e, \tau)\) depends on the working point

\(\tau \approx \text{const}\) is for the pressure propagation from the main chamber to the common rail
Control strategy: main idea

Frequent industrial choice for controlling rail pressure:
PI tuned by heuristic rules plus simple gain scheduling

Several working conditions (equilibria) ⇒ $(K_e, T_e, \tau)$ change ⇒ gain scheduling: a different controller for each working point to keep rail pressure close to reference value, depending on the working conditions

New strategy: fractional-order PI & a new scheduling technique
- a FOPI for each point increases robustness/performance
- gain scheduling to switch between FOPI controllers based on sensitivity of model parameters
Scheduling technique to switch controllers

For each tank pressure, the working point is determined by reference rail pressure and by average duration of injection

Step variations of such variables from initial to final working point:

1. **SMALL VARIATIONS**, bounded by 2 bar and 6 ms: 1 controller designed with reference to \((K_e, T_e, \tau)\) of the final point

2. **LARGE VARIATIONS**: intermediate reference values are considered \(\Rightarrow\) intermediate \((K_e, T_e, \tau)\) \(\Rightarrow\) intermediate controllers

A smooth transition between close working points so that each controller guarantees stability in the neighbors of its working point
Some simulation tests

- Use of a **nonlinear model** implemented by AMESim virtual prototyping tool:
  - properly describes complex fluid dynamic phenomena characterizing injection at different working points
  - model very close to real hardware
- Typical working conditions (rail pressure, injection timing)
- Each yields a different \((K_p, T_i, \tau)\), then a different controller
- FOPI controllers compared to PI controllers
  
  \[ \nu = 1.3, 1.4, 1.5, 1.6 \]
  
  \( \nu < 1.3 \) or \( \nu > 1.6 \) give too high or low phase margins

*Important*: to limit overshoot in \( p_{rail} \) because too much fuel alters the air-fuel ratio and increases consumptions and emissions
First test: small step variation in reference pressure

Engine speed: 2500 rpm  
Tank pressure: 50 bar  
Injectors exciting time: 5 ms

$OS\%$ increases with $\nu$ but much less than with PI: with PI inaccurate metering of the injected fuel!

PI: large variations of control signal (PWM of the valve command) cause high-frequency oscillations and saturation in the control signal (closure of valve) determines large oscillations

FOPI controllers are less sensitive to disturbances and nonlinearities in injection
Second test: large variation in reference pressure

Intermediate reference pressures and switch between 3 FOPI/PI controllers

FOPI yield better and smoother responses with reduced overshoots.

Nonlinearities considerably affect performance of PI.

Other tests made: disturbance (speed variation).
References:


A numerical assignment

Assignment.

Use Matlab to find the CFE-based approximation of order $N = 5$ to $s^{0.5}$ with:

$p_{N,j} = q_{N,N-j} = (-1)^j \binom{N}{j} (\alpha + j + 1)(\alpha - N)_j$

$(\alpha + j + 1)(N-j) = (\alpha + j + 1)(\alpha + j + 2) \ldots (\alpha + N)$

$(\alpha - N)_j = (\alpha - N)(\alpha - N + 1) \ldots (\alpha - N + j - 1)$

$\gg \ldots$
CFE approximation of $s^{0.5}$: Solution for $N = 5$

\begin{align*}
    j = 0 &: (\alpha + 1)_{(5)} = (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5) \\
    j = 1 &: (\alpha + 2)_{(4)} = (\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5) \\
    j = 2 &: (\alpha + 3)_{(3)} = (\alpha + 3)(\alpha + 4)(\alpha + 5) = \\
    j = 3 &: (\alpha + 4)_{(2)} = (\alpha + 4)(\alpha + 5) \\
    j = 4 &: (\alpha + 5)_{(1)} = (\alpha + 5) \\
    j = 5 &: (\alpha + 6)_{(0)} = 1 \\
    j = 0 &: (\alpha - 5)_{(0)} = 1 \\
    j = 1 &: (\alpha - 5)_{(1)} = (\alpha - 5) \\
    j = 2 &: (\alpha - 5)_{(2)} = (\alpha - 5)(\alpha - 4) \\
    j = 3 &: (\alpha - 5)_{(3)} = (\alpha - 5)(\alpha - 4)(\alpha - 3) \\
    j = 4 &: (\alpha - 5)_{(4)} = (\alpha - 5)(\alpha - 4)(\alpha - 3)(\alpha - 2) \\
    j = 5 &: (\alpha - 5)_{(5)} = (\alpha - 5)(\alpha - 4)(\alpha - 3)(\alpha - 2)(\alpha - 1) \\

    p_{5,0} &= (-1)^0 \binom{5}{0} (\alpha + 1)_{(5)} (\alpha - 5)_{(0)} = (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5) \\
    p_{5,1} &= (-1)^1 \binom{5}{1} (\alpha + 2)_{(4)} (\alpha - 5)_{(1)} = 5(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)(5 - \alpha) \\
    p_{5,2} &= (-1)^2 \binom{5}{2} (\alpha + 3)_{(3)} (\alpha - 5)_{(2)} = 10(\alpha + 3)(\alpha + 4)(\alpha + 5)(5 - \alpha)(4 - \alpha) \\
    p_{5,3} &= (-1)^3 \binom{5}{3} (\alpha + 4)_{(2)} (\alpha - 5)_{(3)} = 10(\alpha + 4)(\alpha + 5)(5 - \alpha)(4 - \alpha)(3 - \alpha) \\
    p_{5,4} &= (-1)^4 \binom{5}{4} (\alpha + 5)_{(1)} (\alpha - 5)_{(4)} = 5(\alpha + 5)(5 - \alpha)(4 - \alpha)(3 - \alpha)(2 - \alpha) \\
    p_{5,5} &= (-1)^5 \binom{5}{5} (\alpha + 5)_{(0)} (\alpha - 5)_{(5)} = (5 - \alpha)(4 - \alpha)(3 - \alpha)(2 - \alpha)(1 - \alpha)
\end{align*}
CFE approximation of $s^{0.5}$: Solution for $N = 5$

Replacing $\alpha = 0.5$:

\[
\begin{align*}
    p_{5,0} &= q_{5,5} = 324.84375 \\
    p_{5,1} &= q_{5,4} = 4872.65625 \\
    p_{5,2} &= q_{5,3} = 13643.4375 \\
    p_{5,3} &= q_{5,2} = 9745.3125 \\
    p_{5,4} &= q_{5,1} = 1624.21875 \\
    p_{5,5} &= q_{5,0} = 29.53125
\end{align*}
\]
Assignment. Use Matlab to design $\text{PI}^\alpha$ controller by SOM for a system with:

$K_e = 0.935, \ T_e = 0.124$

and use approximation of order $N = 5$

Verify performance and robustness

>> ...
Assignment for controller design by LS (CASE 1)

Assignment.

Use Matlab to design FOPI controller by LS for a system with:

\[ K_e = 0.9779, \quad T_e = 0.0798 \text{ s} \]

Use \( u_B = 0.3, \quad PM_s = 63^\circ \)

Use approximation of order \( N = 5 \)

Verify performance and robustness

>> ...
Summary of LS design procedure for CASE 1

\[ G_p(s) = \frac{K_e}{s(1 + T_e s)} \quad G_c(s) = K_P + \frac{K_I}{s^\nu} \quad 0 < \nu < 1 \]

1. Set \( \nu \). Specifications: \( u_B \), \( PM_s \). Then: \( u_C \) and

\[ \nu = 1 - \frac{PM_s}{\pi/2} \]

2. Set \( T_I \):

\[ T_I = \frac{(T_e)^\nu (u_C)^{(1-\nu)}}{\sin(0.5 \nu \pi) - u_C \cos(0.5 \nu \pi)} \]

3. Set \( K_I \):

\[ K_I = \frac{1}{K_e} \left( \frac{u_C}{T_e} \right)^{(1+\nu)} \frac{\sin(0.5 \nu \pi) - u_C \cos(0.5 \nu \pi)}{\sin(0.5 \nu \pi)} \]

Set \( K_P = T_I K_I \)

4. Approximate \( s^\nu \) then \( G_c(s) = K_P + \frac{K_I}{s^\nu} \)
Conclusions

- Fractional order (PI) controllers designed by criteria inspired to PI tuning rules (SOM) that are wide-spread in industry

- Loop shaping is for achieving robustness (flat phase plot) to gain changes and good performance (bandwidth for optimal system)
  - Phase plot nearly flat around gain crossover frequency in a sufficiently wide range
  - Design by analytic formulas expressed in closed-form
  - Realization by easy-to-compute, efficient, analytic formulas expressed in closed-form. Stability, minimum-phase and interlacing properties. Accuracy with reduced order of realization. Reduced implementation problems due to finite accuracy.

- Generalizing a widely used criterion for PI could make FOPI controllers be more accepted in industry